MJoSHT
Malaysian Journal of Science, Health & Technology
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Article

Quantum Entanglement and Its Classification Protocols

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Article Info
Article history:
Received: 7 March 2024
Accepted: 25 September2024
Published:7 October 2024
Academic Editor:
Muhammad Safwan Ibrahim

Malaysian Journal of Science, Health & Technology

MJoSHT2024, Volume 10, Issue No. 2
eISSN: 2601-0003

https://doi.org/10.33102/mjosht.v10i2.408

Abstract— **Quantum entanglement has its own major role in quantum information theory. Its application in numerous areas namely quantum computing, quantum cryptography and quantum teleportation are proven vital and essential. Over the last decades, the interests in quantum entanglement have grown and significant progress in quantum computing has been revealed. However, the classification of entanglement was proved to be challenging especially in a higher qubit-dimensional system setting. In this review, indexed literature as the secondary resource was chosen by specific keywords from several database. In reference to the literatures review, there exists several entanglement classifications protocols that will be presented in this paper namely local unitary (LU), local operations and classical communication (LOCC), and stochastic local operations and classical communication (SLOCC). This study will offer a better understanding of quantum entanglement and the entanglement classification protocols.**

Keywords— **Quantum entanglement; Entanglement classification; LU; LOCC ; SLOCC**

I. INTRODUCTION

Quantum entanglement is one of the key assets in quantum information processing. It can be described as a strange phenomenon, an information of an entangled particle can automatically be determined by another entangled particle regardless the distance between those particles, creating non-local interference between subsystems [1, 2]. Quantum entanglement has a major role in various areas namely quantum computing, quantum cryptography and quantum teleportation [3-6]. In quantum computing, the entanglement is the key ingredient to prove the power over its classical counterpart [7-9].

Over the last decades, quantum entanglement has attracted interests of minds and significant progress in quantum computing has been achieved [10, 11]. For instance, in mid-2021 IBM Corporation revealed the 127-qubit (quantum bit) Eagle quantum computing processor chip [12]. This significant progress was then followed by a 433-qubit quantum processer in end of 2022 and breaking the 1000-qubit quantum computing processor chip barrier by 2023 [13, 14]. In 2024, it is predicted that it will be the year that quantum computing will definitively overcome classical computing [15, 16]. Hence, understanding the concept and attributes of quantum entanglement is essential for advancing various quantum technologies in the future.

Even though quantum entanglement itself has been studied for decades, the understanding of the subject matter is considerably low especially in the case of higher pure and mixed qubit-dimensional system [17, 18]. In quantum information processing, entanglement classification is an important process [19-21]. Entanglement classification helps in recognizing the classes and the entanglement level of the entangled particles. Some well-known protocols in entanglement classification are local unitary (LU), local operations and classical communication (LOCC) and stochastic local operations and classical operation (SLOCC) [21, 22]. Entanglement classification is proven to be a challenging and complicated task especially in higher qubit-dimensional system n-qubit ≥ 4 [19, 23].

This review aims to offer a grasp of quantum entanglement as a core in quantum information processing and the existing entanglement classification protocols. This paper is organized as follows. In section 2, the methodology used in selecting literatures for this paper is described. Section 3 narrates the notion of quantum entanglement, and the entanglement classification protocols. Finally, section 4 is devoted to the conclusion.

II. MATERIALS AND METHODS

In this paper, indexed literatures from 1935 to 2022 as secondary resources were selected from several databases namely Scopus, WOS, Google Scholar and other form of publications. Specific keywords were applied such as *quantum entanglement* and *entanglement classification* in the searching process. 44 literatures were selected for this study comprised of the concept of quantum entanglement and entanglement classification. A documentary analysis was performed in Microsoft Word 2021. Fig. 1 depicts the research methodology of this study.

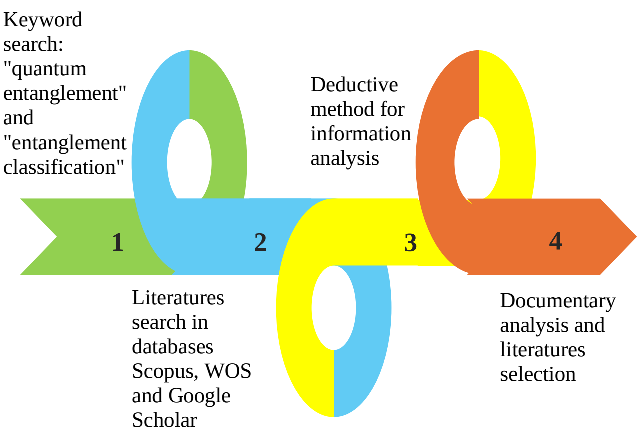

Fig. 1 Research methodology

III. RESULTS AND DISCUSSION

This section discusses the concept of quantum entanglement and the entanglement classification protocols based on previous studies.

A. *Quantum Entanglement*

A "spooky" physical phenomenon known as Einstein-Podolsky-Rosen (EPR) Paradox was first introduced in mid-1930s [24]. Later the phenomenon was referred by Schrödinger in his letter as "*verschränkung*" or an entanglement describing the correlation and interaction between two particles in an experiment [25]. Even so, the concept of the entanglement was not truly comprehended by experts until decades later, Bell proved that the entanglement violates the principles of classical mechanics and only observes the principles of quantum mechanics [26, 27]. Fig. 2 portrayed an entanglement between two particles.

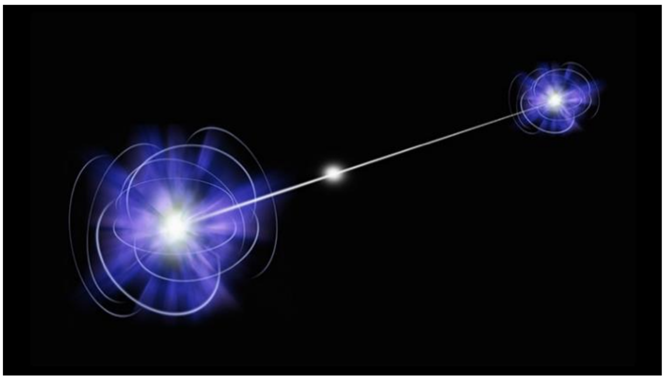

Fig. 2 Quantum entanglement (Source: Castelvecchi, 2020 [28])

Quantum entanglement is represented by a nonfactorizable tensor product of quantum bits or qubits, denoted by $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding to different Hilbert spaces in the case of pure states [1]. In other quantum technology areas, entanglement may occur within systems made up of particles, photons or electrons [29]. A quantum bit or qubit is utterly unique compared to its classical counterpart known as bit, due to its peculiar ability to accommodate 0 and 1 as a value at once [4, 30]. This event is termed as superposition, denoted as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers representing the probability amplitudes being in the state of $|0\rangle$ or $|1\rangle$, a fundamental principle of quantum mechanics [31-33]. Obeying the principle of quantum mechanics, the quantum state of a system is oftentimes expressed in a density matrix form, enabling measurements to be performed upon the system. Beyond the measurements, it is also comprehensively describes the behaviour of the systems in various context, including entanglement and decoherence.

A density matrix of an operator ρ , is a tensor product that can be denoted as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|; \quad 0 \le p_i \le 1; \quad \sum_i p_i = 1 \quad (1)$$

with $|\psi_i\rangle$ is the quantum state and p_i is the probability. A density matrix can represent both pure and mixed quantum state. The equation (1) represents a pure quantum state where $p_i = 1$, while if $p_i \neq 1$ represents a mixed quantum state. A mixed quantum state occurred when a particular state preparation of a system is not completely known, and the entangled systems could not be described by a pure state. In general, a density matrix ρ should satisfies the following properties displayed in the Table 1.

Table I . Properties Satisfied By Density Matrix ρ

Properties	Descriptions
$\rho = \rho^\dagger$	Hermitian matrix; Complex density matrix is equals to its conjugate transpose
$Tr(\rho) = 1$	Trace value for any density matrix must be equals to one
$\rho^2 = \rho,\ Tr\left(\rho^2\right) = 1$	Trace value for pure state is equals to one
$\rho^2 = \rho,\ Tr\left(\rho^2\right) < 1$	Trace value for mixed state is less than one
$0 \le \lambda_i \le 1$	Eigenvalue of density matrix ρ is between 0 and 1
$\langle A\rangle = tr(\rho A)$	The expectation value of the measurement for mixed state can be calculated by this formula, where $\langle A\rangle$ is the expectation value, ρ is the density matrix representing the mixed state and A is the observable

In a bipartite entanglement system, the particles may be entangled even if the distance between both particles is great. For a pure quantum state system, the tensor product between $|\psi\rangle_A$ and $|\psi\rangle_B$ of the bipartite entanglement system $|\psi\rangle_{AB}$ in the Hilbert space may be denoted as

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B \quad (2)$$

whilst in a fully separable form, the bipartite entanglement system $|\psi\rangle_{AB}$ may be defined as

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right) \quad (3)$$

whilst in a Bell state form, the bipartite entanglement system $|\psi\rangle_{AB}$ may be defined as

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \quad (4)$$

For a mixed quantum state system, it is represented as a density matrix ρ , denoted as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \ , \sum_i p_i = 1,\ p_i \ge 0 \quad (5)$$

with p_i is the probabilities. A mixed bipartite entanglement system is fully separable if and only if it can be described as in (2), otherwise it is entangled.

A pure tripartite entanglement system in a Hilbert space, $|\psi\rangle \in H = H_{A_1} \otimes H_{A_2} \otimes H_{A_3}$ involves three correlated particles $|\psi\rangle_A$, $|\psi\rangle_B$ and $|\psi\rangle_C$ in the system. The tensor product of the tripartite entanglement system $|\psi\rangle_{ABC}$ may be denoted as

$$|\psi\rangle_{ABC} = |\psi\rangle_A \otimes |\psi\rangle_B \otimes |\psi\rangle_C \quad (6)$$

it is a fully separable if and only if $|\psi\rangle_{ABC}$ can be described as in (6), otherwise it is entangled. Multiple studies have concluded that in a tripartite entanglement, there are 6 classes under SLOCC; fully separable state, bi-separable state and genuinely entangled state [19, 34-37]. A fully separable state may be described as

$$|\psi\rangle_{ABC} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C \quad (7)$$

with $|\alpha\rangle_A \in H_A$, $|\beta\rangle_B \in H_B$ and $|\gamma\rangle_C \in H_C$. As in bi-separable state, there are three sub-classes denoted as

$$\begin{aligned} |\psi\rangle_{A-BC} &= |\alpha\rangle_A \otimes |\delta\rangle_{BC} \\ |\psi\rangle_{B-AC} &= |\alpha\rangle_B \otimes |\delta\rangle_{AC} \\ |\psi\rangle_{C-AB} &= |\alpha\rangle_C \otimes |\delta\rangle_{AB} \end{aligned} \quad (8)$$

with $|\delta\rangle$ represents two entangled systems and $|\psi\rangle_{A-BC} \in \{H_A, H_B \otimes H_C\}$, $|\psi\rangle_{B-AC} \in \{H_B, H_A \otimes H_C\}$ and $|\psi\rangle_{C-AB} \in \{H_C, H_A \otimes H_B\}$. For genuine entangled state, there are Greenberger-Horne-Zeilinger (*GHZ*) and *W* state, both denoted as

$$\begin{aligned} |GHZ\rangle &= \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right), \\ |W\rangle &= \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \end{aligned} \quad (9)$$

In a Hilbert space $|\psi\rangle \in H = H_{A_1} \otimes H_{A_2} \otimes H_{A_3}$, the mixed state, ρ is fully-separable if and only if ρ may be described by

$$\rho = \sum_{i=1}^{k} p_i \rho_{A_1}^i \otimes \cdots \otimes \rho_{A_n}^i \quad (10)$$

with $\sum_{i=1}^{k} p_i = 1$, where p_i is probability of each mixed state, otherwise it is entangled. A fully separable mixed state may be described as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad (11)$$

with p_i is convex combination weights also known as mixing weights and $|\psi_i\rangle$ is fully separable state. As for bi-separable mixed state, it is denoted as in (11) with $|\psi_i\rangle$ is bi-separable state with a non-completely positive map. Briefly, the non-completely positive map in quantum mechanics is a type of quantum channel that fails to preserve positivity when extended to larger quantum systems. This means that when applied to composite quantum system, the map can lead to negative values in the resulting density matrix, violating the fundamental requirement of quantum mechanics, which is positivity. For genuinely entangled state, it is also denoted as in (11) with $|\psi_i\rangle$ is genuinely entangled state.

B. *Entanglement Classification Protocols*

Entanglement classification is important in quantum mechanics [21, 34]. It determines the quantum state types or classes of the entangled system. Numerous entanglement classification methods has been introduced but even so the understanding of the subject matter is still considerably little [38]. Entanglement classification for bipartite or two-qubit entanglement system has been considered settled, as the quantum states can either be classified into fully separable or entangled due to its nature involving only two quantum systems [22, 39-41].

Most well-known entanglement classification protocols are local unitary (LU), local operations and classical communication (LOCC) and stochastic local operations and classical communication (SLOCC). These protocols to some extend are correlated with each other. Fig. 3 depicts the relationship between LU, LOCC and SLOCC.

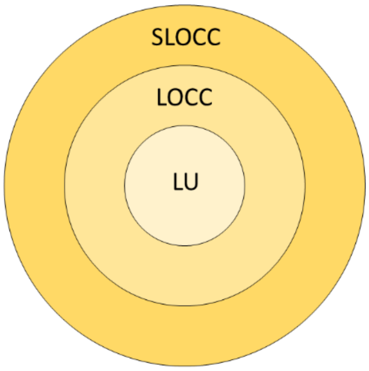

Fig. 3 The relationship between LU, LOCC and SLOCC (Source: Adapted from “Designing locally maximally entangled quantum states with arbitrary local symmetries”, Słowik, Sawicki [42]).

It is prominent in quantum information theory that as the number of n-qubit grows, the process of classifying entanglement becomes challenging [40]. Collective studies concluded that entanglement classification for two and three-qubit pure entanglement system under SLOCC has completed [21, 41]. It was determined that there are six entanglement classes under SLOCC for three-qubit entanglement system. However, for n-qubit ≥ 4 and higher-dimensional system, it is unclear as there is no conclusive decision on how many entanglement classes and families there are, made it a challenging task to classify the quantum system state due to its complex structure [21, 22, 36, 37, 41, 43-45].

Local unitary operations may be denoted in a form of $U = U_A \otimes U_B$ where U_A and U_B are unitary operators independently act and do not change operations between subsystems [46, 47]. The product state is preserved under local unitary (LU). Local unitary classification for a bipartite system can be straightforwardly done using Schmidt decomposition [48-50]. Nevertheless, it is a different story for n-qubit ≥ 3 system. Two entangled systems are LU-equivalent if both are convertible into each other via local operations [40, 51, 52]. In addition, according to [21, 40, 51, 52], a LU-equivalent entangled systems possess the same amount of entanglement, which means it can be used to perform different given tasks.

In LOCC, two entangled systems executing the same task are equivalent if its state can surely be obtained from both systems [47, 48]. Additionally, entanglement do not increase under LOCC [40, 53, 54]. Hence, the LOCC protocol may signify different equivalence classes of quantum states. On top of that, LOCC may unveil the distinctive nonlocal features of the quantum states [55]. As illustrated in Figure 3, according to [47], any quantum systems interconvertible by local unitary (LU) is definitely a LOCC equivalent. Additionally, if the interconvertibility feature is discarded and substituted with invertible, then the LOCC turns into the SLOCC [21, 45, 48].

As reported in [56], a natural way in classifying entanglement is produced by any two states that are convertible under SLOCC. The protocol is built on a sequence of local operations, which the information gained from it can be used for further local operations. A multiqubit state that can be acquired with nonzero probability from another multiqubit state utilizing the local invertible operations are SLOCC equivalent [57-59]. Additionally, multipartite pure states are equivalent under SLOCC and shares the same entanglement structure [38, 60]. Under the protocol, any two states that can be acquired by each other are suitable for performing the same tasks [61]. As mentioned, there are approximately six different classes of entanglement under SLOCC protocol. Two separable, two biseparable, and two genuinely entangled classes. In a tripartite pure system, there are only GHZ and W state exists as a genuinely entangled states [19]. A GHZ state is known as the genuine tripartite entanglement whilst W state is known to have the maximal amount of bipartite entanglement if a system is traced. Additionally, any state may be converted into either GHZ state or W state under SLOCC [41].

Entanglement classification via LU protocol was demonstrated in [21, 48, 51, 52]. While via LOCC, it was demonstrated in [45, 62-64]. Lastly via SLOCC, it was demonstrated in [21, 45, 57, 65]. In the case of three-qubit entanglement system, it can be classified by numerous methods. One of the most known methods yet under-studied, that will be discussed in the next section is special linear group (SL).

C. Special Linear Group for Entanglement Classification

The special linear group, *SL*, a subgroup of the general linear group $GL(n,F)$, where F is the field of $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers) [66]. The special linear group, *SL* denoted as $SL(n,\mathbb{C})$ of degree n is the set of $n\times n$ matrices with the value 1 as the determinant and is a genuine Lie group [67-70]. The unique mathematical properties and its application of the special linear group in distinguishing between different entangled quantum states is presented. Here, a set of

$$G=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{C}, ad-bc=1\right\} \quad (12)$$

is presented. For a set to be a group, it must satisfy these conditions:

a. Closed Operation.

A set $(G,*)$ with an operation $*$, for any two elements of $a,b\in G$ associates with an element $x=a*b\in G$. For example, in $(X,+)$, $2+4=6$, $6+(-4)=2$.

b. Existence of an identity element.

For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This shows that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element of G.

c. Existence of inverse.

Given any $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$, its inverse is $A^{-1}=\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\in G$. To prove, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This shows that $A^{-1}=\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\in G$ is the inverse element.

d. Associativity.

Note that a matrix multiplication (*) is associative, and for $A,B\in G$, the determinant of *A* must be, $\det(A)=1$ and determinant of *B* must be, $\det(B)=1$ [66]. Then to prove that $A,B\in G$, the product of determinant of *A* and *B*, $\det(AB)=\det A.\det B=1.1=1$. Therefore $A,B\in G$ and matrix multiplication (*) is associative.

This shows that set $(G,*)$ is a group, precisely a special linear group. Mathematically, special linear group, $SL(n,\mathbb{C})$ can be represented as [57, 71]

$$SL(\mathbb{C}^{d_1})\times\ldots\times SL(\mathbb{C}^{d_N}) \quad (13)$$

In quantum mechanics, entanglement in quantum states is typically characterized by mathematical construct, such as entanglement measures or invariants under the $SL(2,\mathbb{C})$ and related groups. For example, in the case of a single qubit described by a 2×2 density matrix ρ, the determinant of ρ reflects properties like purity, but does not directly indicate entanglement. For two qubits on the other hand, entanglement can be analysed using quantities such as concurrence or negativity, which are derived from the density matrix and its partial transpose, rather than directly from the determinant of any $SL(2,\mathbb{C})$ matrix. The $SL(2,\mathbb{C})$ matrices, characterized by a determinant value of 1, play a crucial role in entanglement classification due to their unique properties. These matrices are used to describe the behavior of quantum states, distinguishing between entangled and separable states. The $SL(2,\mathbb{C})$ matrices are essential in highlighting the quantum connections within each quantum systems, and can be represented as

$$SL(2,\mathbb{C}):=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{C}, ad-bc=1\right\} \quad (14)$$

The special linear group is a non-abelian or also known as non-commutative group [72]. A mathematical operation is abelian or commutative if the order of operands changed, but the result is unaffected. Here, $SL_q(2,\mathbb{C})$ is the definition of $SL(2,\mathbb{C})$ where q is the parameter. A matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to general linear group $GL_q(2,\mathbb{C})$. According to [72], matrix A has a following distinct property, its non-commutative matrix elements satisfying the commutation relations

$$\begin{aligned} &ab=qba \\ &ac=qca, \quad bc=cb \\ &bd=qdb, \quad ad-da=\left(q-\frac{1}{q}\right)bc \\ &cd=qdc \end{aligned} \tag{15}$$

Let

$$A'=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{16}$$

be the same type of matrix, its elements satisfy commutative elements similarly as in (15)

$$\begin{aligned} &a'b'=qb'a' \\ &a'c'=qc'a', \quad b'c'=c'b' \\ &b'd'=qd'b', \quad a'd'-d'a'=\left(q-\frac{1}{q}\right)b'c' \\ &c'd'=qd'c' \end{aligned} \tag{17}$$

Let a' , b' , c' , d' commute with a , b , c , d . Thus, the matrix $A''=AA'$

$$A''=\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}=\begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix} \tag{18}$$

be the same type of matrix, its elements satisfy,

$$\begin{aligned} &a''b''=qb''a'' \\ &a''c''=qc''a'', \quad b''c''=c''b'' \\ &b''d''=qd''b'', \quad a''d''-d''a''=\left(q-\frac{1}{q}\right)b''c'' \\ &c''d''=qd''c'' \end{aligned} \tag{19}$$

In accordance with matrices A , A' and A'' obtained in the limit $q\to 1$ when its elements commute, the classical operation turns into quantum operation. In physics, it is known as a mathematical quantization, a systematic transition method of constructing quantum mechanics from its classical form [73, 74].

IV. CONCLUSIONS

As a conclusion, a review of the concept of quantum entanglement and the entanglement classification protocols was presented. It is crucial to understand the fundamentals of quantum entanglement as it has a significant role in the advancing of quantum technologies such as quantum computing. Aside from understanding the main concept, entanglement classification is also regarded as an important process and valuable to comprehend as it aids in determining the entanglement classes particles belongs to and the level of entanglement of entangled particles.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

ACKNOWLEDGEMENT

This paper is part of research project supported by the Malaysian Ministry of Higher Education Fundamental Research Grant Nos. FRGS/1/2021/ICT04/USIM/02/1 (USIM/FRGS/KGI/KPT/50121).

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