

Article

## Analysis of Different Boundary Conditions on Homogeneous One-Dimensional Heat Equation

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**Abstract**— Partial differential equations involve results of unknown functions when there are multiple independent variables. There is a need for analytical solutions to ensure partial differential equations could be solved accurately. Thus, these partial differential equations could be solved using the right initial and boundaries conditions. In this light, boundary conditions depend on the general solution; the partial differential equations should present particular solutions when paired with varied boundary conditions. This study analysed the use of variable separation to provide an analytical solution of the homogeneous, one-dimensional heat equation. This study is applied to varied boundary conditions to examine the flow attributes of the heat equation. The solution is verified through different boundary conditions: Dirichlet, Neumann, and mixed-insulated boundary conditions. The initial value was kept constant despite the varied boundary conditions. There are two significant findings in this study. First, the temperature profile changes are influenced by the boundary conditions, and that the boundary conditions are dependent on the heat equation's flow attributes.

**Keywords**— Separation of variables; one-dimensional flow; homogeneous heat equation; boundary conditions; analytical solution.

### I. INTRODUCTION

Partial differential equations involve the results of undetermined functions with multiple independent variables. Partial differential equations could be divided into three distinct functions: hyperbolic, elliptic (Biala & Jator, 2015; Papanikos & Gousidou-Koutita, 2015), and parabolic (Agyeman & Folsom, 2013; Mamun *et al.*, 2018). The hyperbolic partial differential equation describes the vibrations and transformation in the wave of elastic strings. A parabolic partial differential equation is used in numerous scientific calculations, for instance, determining heat diffusion and propagation in the ocean's acoustics. Lastly, elliptic partial differential equations are applied to describe equations like the Laplace.

Studies have solved partial differential equations through either an analytical or numerical solution (Subani *et al.*, 2020). In this light, studies like Abarbanel *et al.* (2000), Islam *et al.* (2018), Mebrate (2015), and Roknujjaman and Asaduzzaman (2018) advocated that a general understanding of the equation theory could help validate the numerical solution. The use of an analytical solution could ensure the accurate solutions for the equations are achieved.

Studies argued that these equations could only be solved when the accurate boundary and initial conditions are applied. In this case, the boundary conditions are relative to the general solution. Thus, these equations will have unique general

solutions when various sets of boundary conditions are used. For instance, Subani *et al.* (2020) only considered the Neumann boundary in the heat equation and found that the heat temperature quickly converges to zero when a short rod is used.

The heat equation proliferates the highly non-physical energy at an infinite speed. On the other hand, the heat equation is still highly regarded as a valid model of temperature evolution in engineering and physics equations. Heat transfer causes temperature changes, as heating increases the temperature while cooling decreases the temperature (Tveito & Winther, 1998). Crank (1975) argued that it ensures no work is done on or by the system and that no step shift in this process occurs. Javed (2012) examined the sources of dry heat, specifically electric pads, radiant heat, and hot water bottles, as well as moist sources of heat. The study found that while moist heat seems to be more penetrating than dry heat, moist materials have slower heat loss than dry ones.

Another study by Sabaeian *et al.* (2008) highlighted the crucial role of temperature distribution, from an Islamic perspective, in measuring, simulating and predicting thermal effects. Aidoo and Wilson (2015) emphasised that heat changes are relative to physical or chemical changes. The temperature reflects the amount of energy required to change a substance's temperature or its heat capacity. The Quran describes numerous phenomena related to temperature, for

instance, Surah Yassin, verse 80, which describes how fire sparked from green trees (Al-Mahalli, 2003; As-Suyuti, 2003). In general, the verse explains how green plants could produce fire, specifically through the frictions between two surfaces (Al-Mahalli & As-Suyuti, 2007).

It can be deduced that the friction velocity of the two objects affects the heat velocity rate. Thus, the heat velocity is calculated from the high to low heat zone. The variable was applied to provide an analytical solution for the homogeneous one-dimensional heat equation. This study aims to examine the heat equation's flow characteristics with different boundary conditions. In this regard, the heat equations with the Dirichlet, Neumann, and mixed boundary conditions are solved. The results are then compared with different boundary conditions with similar initial conditions.

## II. MATHEMATICAL FORMULATION

The one-dimensional homogeneous heat boundary value problems with the Dirichlet boundary condition presented below were defined using the mathematical models. The heat equation was considered to calculate the change in the temperature function,  $u$  over time,  $t$ . The physical model of the heat equation problem is shown in Fig. 1 below.

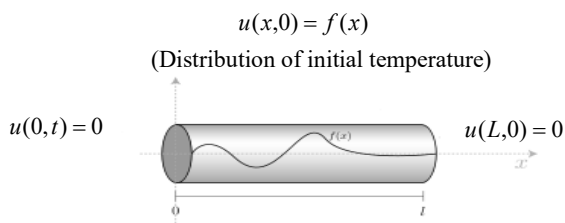


Fig. 1 Physical model of heat equation problem with Dirichlet boundary condition

In this regard, the general, one-dimensional homogeneous heat equation contains  $u$ , which represents the heat temperature,  $x$  which represents space and  $t$ , which represents time, to form Equation 1 below,

$$u_t(x,t) = u_{xx}(x,t), \quad 0 < x < 1, \quad t > 0. \quad (1)$$

### A. Boundary Conditions

TABLE I  
BOUNDARY CONDITIONS FOR THE STARTING POINT AND AT THE ENDPOINT

Boundary Condition	Equation
Dirichlet boundary conditions	$u(0,t) = 0, \quad t > 0$ $u(1,t) = 0, \quad t > 0$ (2a)
Neumann boundary conditions	$u_x(0,t) = 0, \quad t > 0$ $u_x(1,t) = 0, \quad t > 0$ (2b)
Mixed boundary condition	At the rod's left end, insulated $u_x(0,t) = 0, \quad t > 0$ $u(1,t) = 0, \quad t > 0$ (2c)

At the rod's right end insulated	$u(0,t) = 0, \quad t > 0$ $u_x(1,t) = 0, \quad t > 0$ (2d)
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In this study, the Dirichlet, Neumann, and mixed boundary conditions (left end insulated and right end insulated) were considered in solving the heat temperature problems. The equations for the conditions at the starting point ( $x = 0$ ) and the endpoint are presented in Table 1.

### B. Initial Conditions

The conditions at the initial point ( $t = 0$ ) are given as,

$$u(x,0) = x, \quad 0 < x < 1. \quad (3)$$

Equations (1) - (3) define the heat conduction in a one-dimensional, uniform rod. The rod has no internal heat source, and the thermal diffusivity was set to 1; with initial  $x$  at  $0 < x < 1$ , it has perfect lateral insulation.

## III. ANALYTICAL SOLUTION

Studies have numerically solved heat equations using the finite element method (Dabral & Dhawan, 2011; Susan *et al.*, 2008), finite different method (Gerald, 2011; Jalil, 2011; Zana, 2014) and Crank-Nicolson (Emenogu & Oko, 2015). However, we need to provide an analytical solution to obtain accurate solutions for the partial differential equation. Equation (1) was solved using the Separation of Variables (SOV). In this case, the partial differential equation was divided into two differential equations, and 1 independent variable was maintained in each Equation.

### A. Separating the variables using SOV

Equation (1) is expressed as,

$$u(x,t) = X(x)T(t), \quad (4)$$

with

$$u_t(x,t) = X(x)T'(t), \quad (5)$$

$$u_{xx}(x,t) = X''(x)T(t),$$

Equation (5) is substituted in Equation (1) to obtain:

$$X(x)T'(t) = X''(x)T(t),$$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = k, \quad (6)$$

Subsequently, the two differential equations are divided into the  $X$ -problem and  $T$ -problem as shown below,

$$X\text{-problem: } X''(x) = kX(x), \quad (7a)$$

$$T\text{-problem: } T'(t) = kT(t), \quad (7b)$$

**B. Applying the Sturm-Liouville Solver in X-Problem**

The X-problem was solved using the Equation below. In this regard, Equation (1) was resolved using the Dirichlet boundary conditions (2a). The boundary conditions are,

$$u(0,t) = X(0)T(t), \text{ with } X(0) = 0, \quad T(t) \neq 0, \quad (8a)$$

and

$$u(1,t) = X(1)T(t), \text{ with } X(1) = 0, \quad T(t) \neq 0. \quad (8b)$$

Three considerations were made when solving the boundary value problem (BVP),

$$\lambda = 0, \lambda > 0; \lambda < 0, \lambda = -\lambda^2; \lambda \neq 0.$$

Eigenvalues  $\lambda = 0$  and  $\lambda > 0$ , reflect the trivial solution.

$$\lambda < 0, \lambda = -\lambda^2; \lambda \neq 0 \text{ indicate the non-trivial solution.}$$

It is crucial to consider the non-zero (non-trivial) solutions as the Equation's trivial solution is a constant zero solution. In this regard, in the constant  $k = -\lambda$  of separation, the negative sign could either be positive, negative, or zero. Thus, Equation (7a) determines the two-point boundary value problem's eigenvalues and Eigenfunctions, which is expressed as;

$$X''(x) = -\lambda^2 X(x),$$

The characteristic Equation can be expressed as  $m^2 + \lambda^2 = 0$ , to solve the complex conjugate roots  $m = \pm \lambda i$ .

The solution can be expressed as,

$$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x), \quad (9)$$

Thus, the researcher included the (8a) and (8b) in Equation (9) so that,

$$X_n(x) = C_n \sin(n\pi x), \quad (10)$$

with  $X_0(x) = 0, \lambda_n = n\pi$  and  $n = 1, 2, 3, \dots$ .

**C. T-Problem Solver**

The integration of the T-problem yields:

$$T_n(t) = B_n e^{-(n\pi)^2 t}, \quad (11)$$

with  $n = 1, 2, 3, \dots$ .

**D. Fundamental Solution**

The fundamental solution shown below can be obtained by substituting equations (10) and (11) into Equation (4),

$$u_n(x,t) = X_n(x)T_n(t),$$

$$u_n(x,t) = A_n \sin(n\pi x) e^{-(n\pi)^2 t},$$

with  $u_0(x,t) = A_0$  and  $n = 1, 2, 3, \dots$ .

Equation (12) was used to solve the one-dimensional homogeneous heat equation with the Neumann boundary condition,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-(n\pi)^2 t}. \quad (12)$$

**E. Specific Solution**

The researcher integrated Equation (12) into the initial condition (3) to form

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = f(x), \quad (13)$$

with

$$A_n = \frac{2}{L} \int_0^L f(x) X(x) dx,$$

$$A_n = 2 \int_0^1 x \sin(n\pi x) dx = \frac{2}{n\pi} [1 - (-1)^n], \quad (14)$$

where  $n = 1, 2, 3, \dots$ .

Equation (15) presents the full solution for Equation (1),

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (-1)^n] \sin(n\pi x) e^{-(n\pi)^2 t}. \quad (15)$$

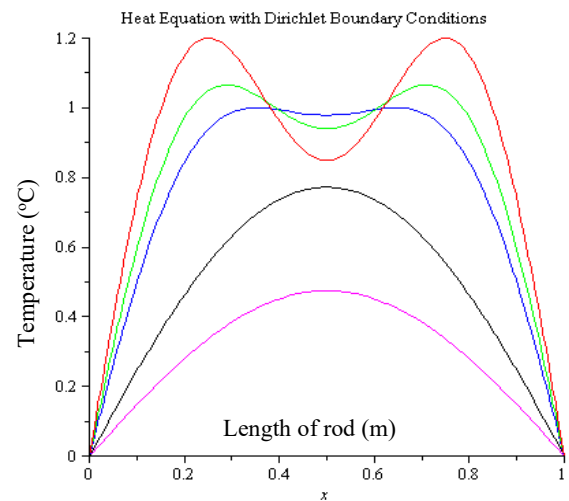
**IV. RESULTS AND DISCUSSION**

TABLE II  
ANALYTICAL SOLUTIONS OF DIFFERENT TYPES OF BOUNDARY CONDITIONS

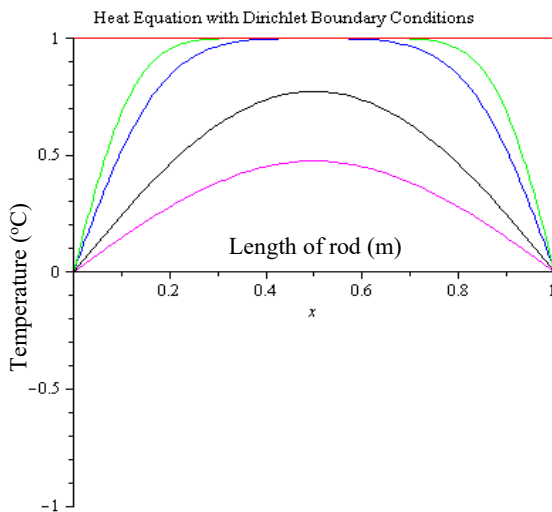
Boundary Condition	Equation
Dirichlet: $u(0,t) = 0, \quad t > 0$ $u(1,t) = 0, \quad t > 0$	$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - (-1)^n] \times \sin(n\pi x) e^{-(n\pi)^2 t}$
Neumann: $u_x(0,t) = 0, \quad t > 0$ $u_x(1,t) = 0, \quad t > 0$	$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} [(-1)^n - 1] \times \cos(n\pi x) e^{-(n\pi)^2 t}$
Right end insulated: $u(0,t) = 0, \quad t > 0$ $u_x(1,t) = 0, \quad t > 0$	$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^2 \pi^2} (-1)^{n+1} \times \sin\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t}$
Left end insulated: $u_x(0,t) = 0, \quad t > 0$ $u(1,t) = 0, \quad t > 0$	$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[ -\frac{4}{(2n-1)\pi} + \frac{8}{(2n-1)^2 \pi^2} \right] \times \cos\left(\frac{(2n-1)\pi x}{2}\right) e^{-\left(\frac{(2n-1)\pi}{2}\right)^2 t}$

Table II presents the analytical solutions for the various boundary conditions with mixed Dirichlet and Neumann boundary conditions. The initial condition was kept constant with  $u(x,0) = x$ ,  $0 < x < 1$ . It was observed that the different boundary conditions resulted in different solutions, despite the same initial conditions used.

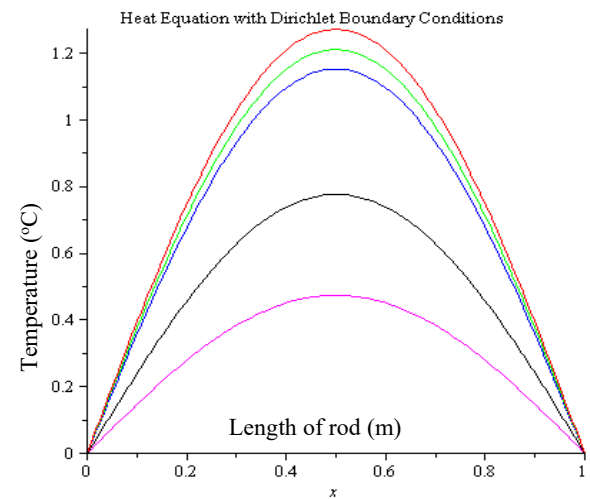
Fig. 2 illustrates the heat profile across the rod with Dirichlet boundary conditions at  $t > 0$ . The duration varied from 0.000s (red line), 0.005s (green line), 0.010s (blue line), 0.050s (black line) 0.100s (magenta line) and 1.000s (maroon line), with different iterations,  $1 \leq n \leq \infty$ . The temperature profiles are valid when  $1 \leq n \leq 2$ , at  $t = 0.000$ s. The temperature started increasing and decreasing after  $n > 3$ , before substantially decreasing at  $n = \infty$  when the rod's length is 1 meter. The temperature at both the rod's left and right ends is  $0^\circ\text{C}$ . Subsequently, at  $x > 0$ , the temperature rises before falling at the middle of the rod. The maximum temperature recorded is  $1.27^\circ\text{C}$ .



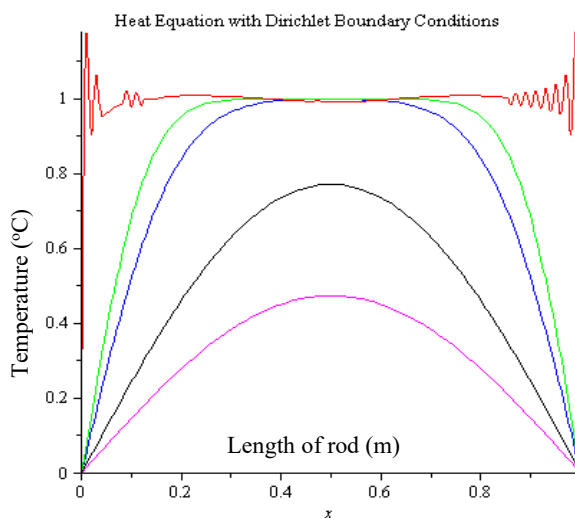
(c)  $3 \leq n \leq 9$



(a)  $n = \infty$



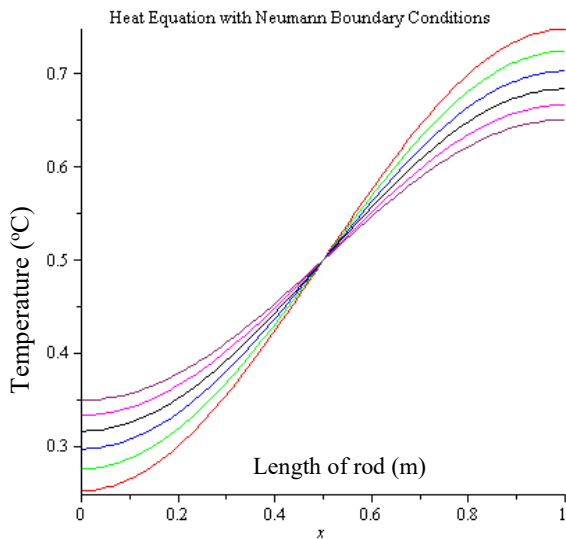
(d)  $1 \leq n \leq 2$



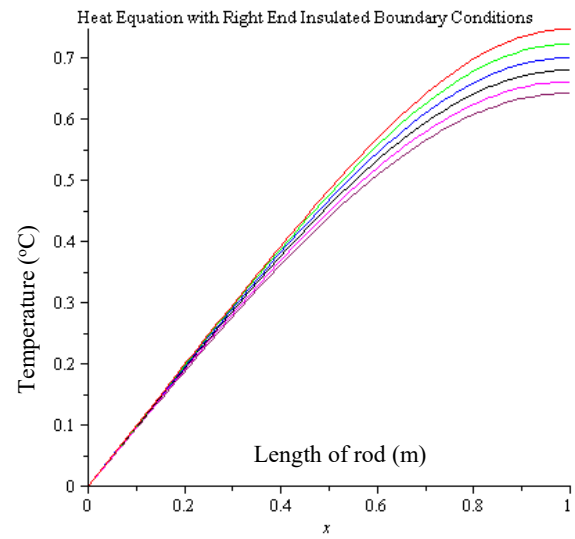
(b)  $n = 1000$

Fig. 2 Temperature profile  $u(x,t)$  across time,  $t$  and number of iteration,  $n$  with Dirichlet boundary conditions

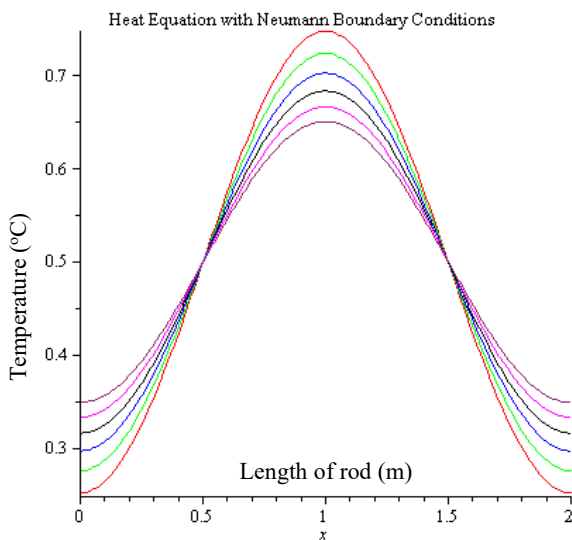
The heat profile across the rod with Neumann boundary conditions at  $t > 0$  is shown in Fig. 3. illustrates. The heat profile was recorded at different time durations, 0.000s (red line), 0.005s (green line), 0.010s (blue line), 0.050s (black line) 0.100s, (magenta line) and 1.000s (maroon line), with different iterations,  $1 \leq n \leq \infty$ . It could be observed that the temperature profile pattern for all iterations  $1 \leq n \leq \infty$ . is similar, indicating that insulating the rod's left and right ends stopped the heat from moving to the end and increasing the temperature. The highest temperature recorded at the end of the rod is  $0.99^\circ\text{C}$ , indicating a temperature drop. The temperature profile presented in Fig. 3 (a) only shows half of the temperature distribution cycle. In this regard, the rod should be lengthened to  $L = 2\text{m}$  so that the complete cycle of temperature distribution could be captured.



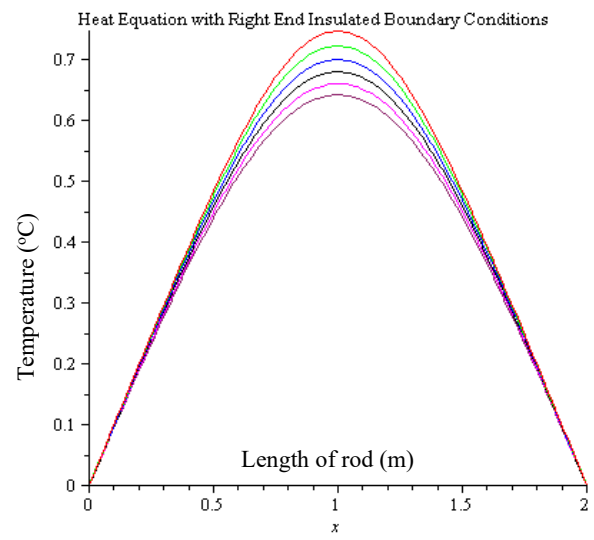
(a) Length of the rod  $L = 1\text{m}$



(a) Length of the rod  $L = 1\text{m}$



(b) Length of the rod  $L = 2\text{m}$



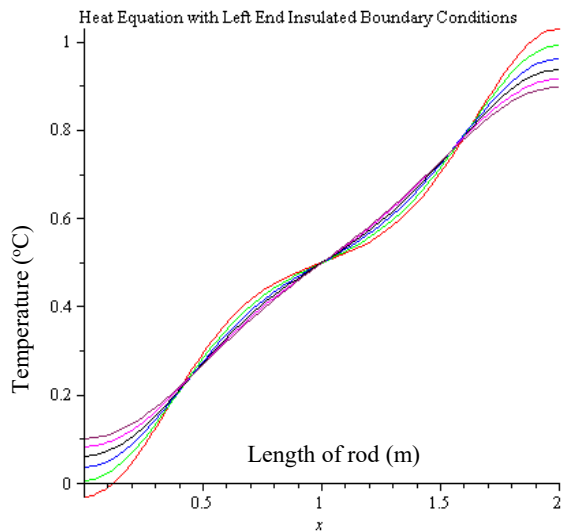
(b) Length of the rod  $L = 2\text{m}$

Fig. 3 Temperature profile  $u(x,t)$  with Neumann boundary conditions for a different time,  $t$ , and number of iteration,  $n$

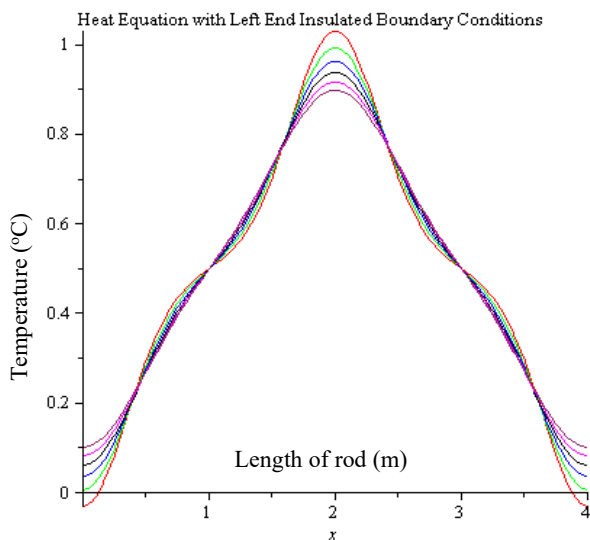
Fig. 4 illustrates the heat profile across the rod with the right end insulated boundary conditions at  $t > 0$ . The profile was recorded at the time durations of 0.000s (red line), 0.005s (green line), 0.010s (blue line), 0.050s (black line), 0.100s (magenta line) and 1.000s (maroon line), with different iterations,  $1 \leq n \leq \infty$ . A similar temperature profile pattern was observed for all iterations  $1 \leq n \leq \infty$ . As shown in Fig. 4(a), with a 1-meter rod, the experiment only captured half the temperature distribution cycle. Subsequently, the rod's length should be increased to 2 meters to capture the full temperature distribution. The maximum temperature recorded at the middle of the rod is 0.99 °C. Subsequently, it dropped to 0 at the rod's end.

Fig. 4 Temperature profile  $u(x,t)$  across time,  $t$  and iterations,  $n$  with insulated right end boundary

The rod's heat profile at  $t > 0$  for the left end insulated boundary conditions is shown in Fig. 5. The profile was recorded at 0.000s (red line), 0.005s (green line), 0.010s (blue line), 0.050s (black line), 0.100s (magenta line) and 1.000s (maroon line), with different iterations,  $1 \leq n \leq \infty$ . A similar temperature profile was recorded for all iterations  $1 \leq n \leq \infty$ . In Fig. 5(a), the rod length is 2m, and only a half-cycle of the temperature distribution in the temperature profile was captured. Thus, the rod needs to be lengthened to  $L = 4\text{m}$  so that the full temperature distribution cycle could be captured. Thus, when the rod's length is 2 meters, the temperature at the middle of the rod is 1.97 °C, and it decreased to zero towards the end.



(a) Length of the rod  $L = 2\text{m}$



(b) Length of the rod  $L = 4\text{m}$

Fig. 5 Temperature profile  $u(x, t)$  across time,  $t$  and iterations,  $n$  with left end insulated boundary conditions

## V. CONCLUSION

There is a need to apply the accurate boundaries and initial conditions in analytical solutions to ensure partial differential equations could be accurately solved. As boundary conditions are relative to the general solution, these equations will have various solutions when different boundary conditions are used. This study presents the analytical solutions for the homogenous one-dimensional heat equation problem using the variable separation process. This study aims to identify the heat equation's flow attributes by considering the mixed, Neumann, and Dirichlet boundary conditions. For this study, the researcher used the same initial value for the three boundary conditions. The temperature reading was noted at 0.000s, 0.005s, 0.010s, 0.050s, 0.100s and 1.000s with different iterations. The temperature profile for Dirichlet

boundary conditions is valid when  $1 \leq n \leq 2$ . at  $t = 0.000\text{s}$ . When a 1-meter rod was used at  $n > 3$ , the temperature rose and fell before it eventually decreased significantly ( $n = \infty$ ). At the rod's left end and right end, the temperature was constant at  $0^\circ\text{C}$  before increasing at  $x > 0$  and reducing again at the rod's half-length. The temperature profile patterns for all iterations  $1 \leq n \leq \infty$  are similar for the Neumann boundary conditions. On the other hand, only half of the temperature distribution was captured for the rod with the mixed, left, and right end insulated boundary conditions when a 1 meter and 2 meters rod is used. From this observation, there is a need to use longer rods ( $L = 2\text{ m}$  and  $L = 4\text{ m}$ ) to capture the full temperature distribution cycle.

## CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

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